Solutions to the tetrahedron and 3D reflection equations from quantum cluster algebras

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1. **Tetrahedron and 3D reflection equations**
2. **Quantum cluster algebra**
3. **Application to the tetrahedron equation**
4. **Relation to known solutions**
5. **Outlook**

Most of the presentation will be about the tetrahedron equation for simplicity.

1. Tetrahedron and 3D reflection equations

(3D analogue of the Yang-Baxter and reflection eqs.)

Tetrahedron eq. [A.B. Zamolodchikov 80]

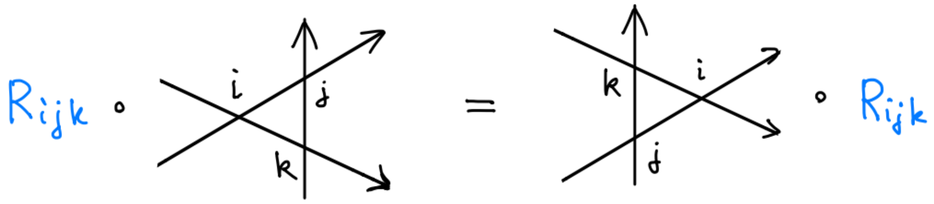
$$R_{124}R_{135}R_{236}R_{456} = R_{456}R_{236}R_{135}R_{124} \ \text{ on } V^{\otimes 6} \qquad R_{ijk} \in \mathrm{End}(\overset{i}{V} \otimes \overset{j}{V} \otimes \overset{k}{V})$$

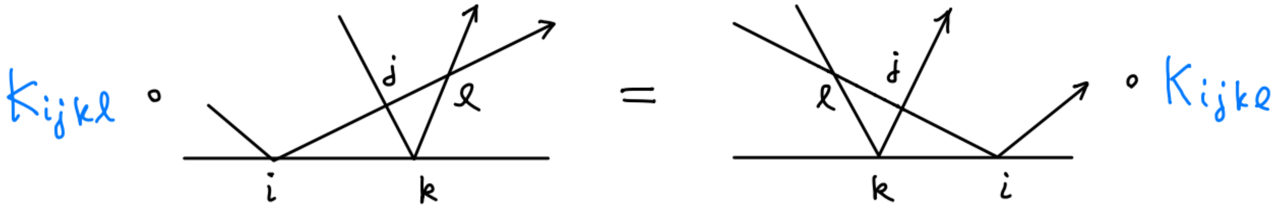
3D reflection eq. [Isaev-Kulish 97]

$$R_{689}K_{3579}R_{249}R_{258}K_{1478}K_{1236}R_{456} = R_{456}K_{1236}K_{1478}R_{258}R_{249}K_{3579}R_{689}$$

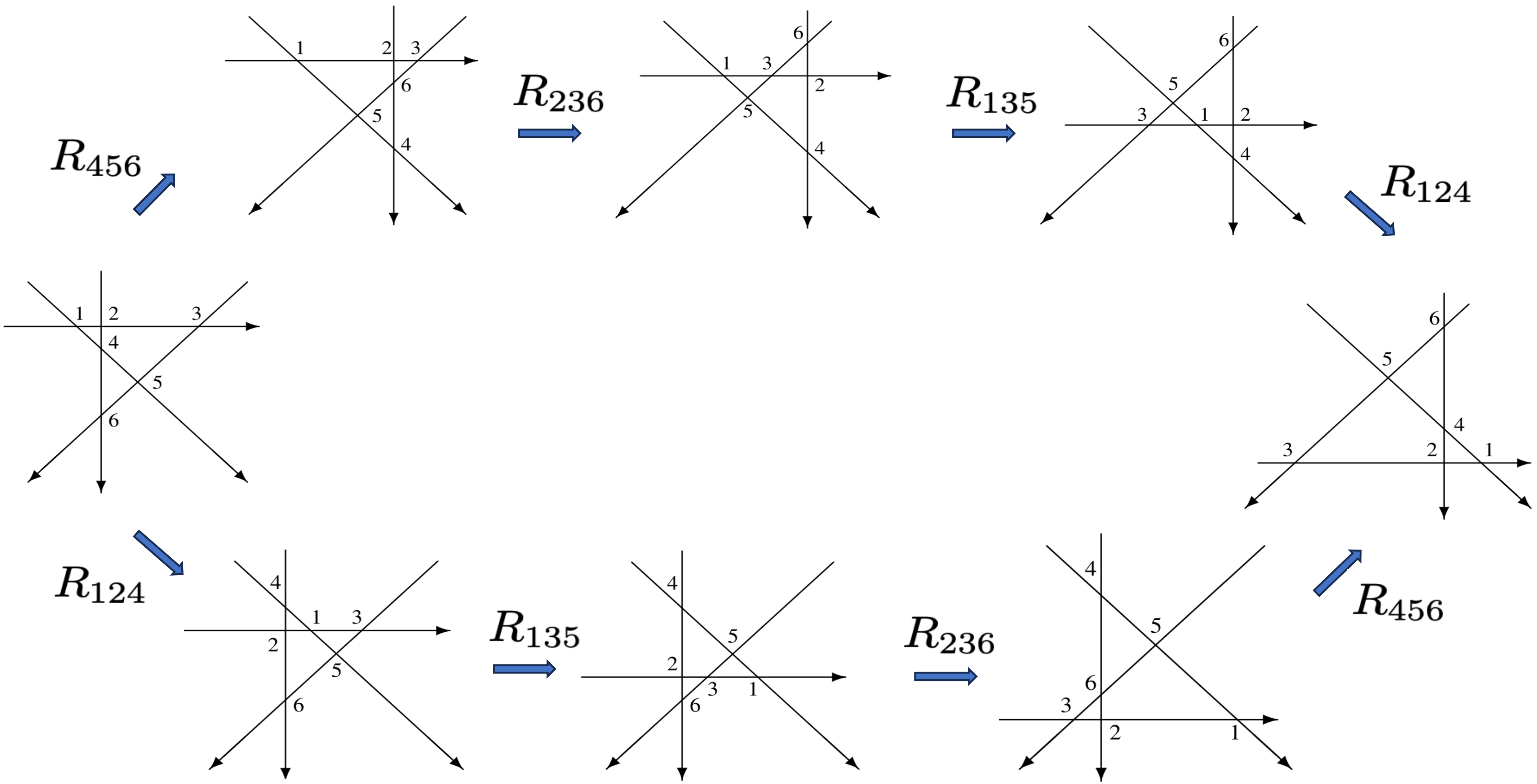
$$\text{on } W \otimes V \otimes W \otimes V \otimes V \otimes V \otimes W \otimes V \otimes V \qquad K_{ijkl} \in \mathrm{End}(\overset{i}{W} \otimes \overset{j}{V} \otimes \overset{k}{W} \otimes \overset{j}{V})$$

They are compatibility conditions of the quantized Yang-Baxter eq. and quantized reflection eq., which are the *usual* Yang-Baxter and reflection equations up to conjugation.



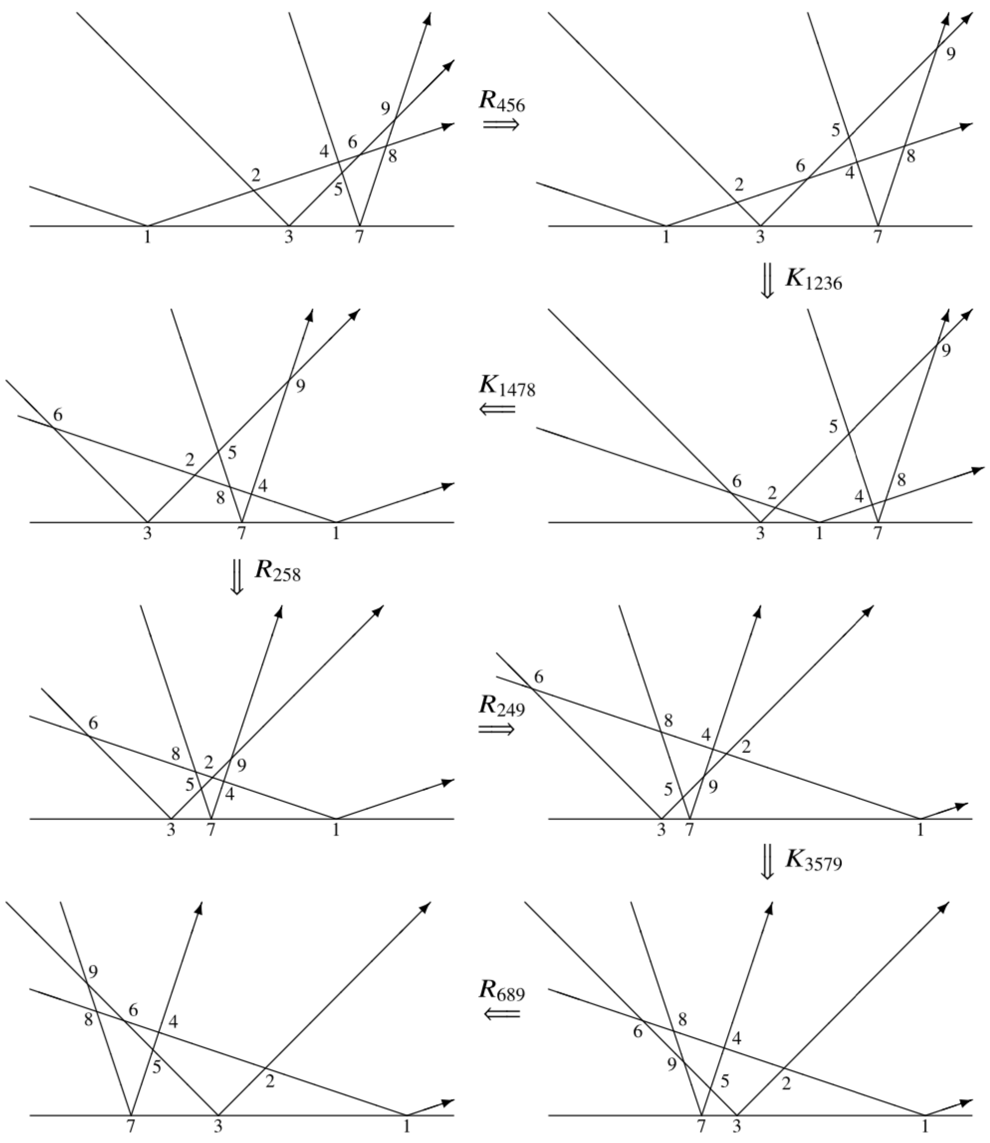


$$R_{124}R_{135}R_{236}R_{456} = R_{456}R_{236}R_{135}R_{124}$$

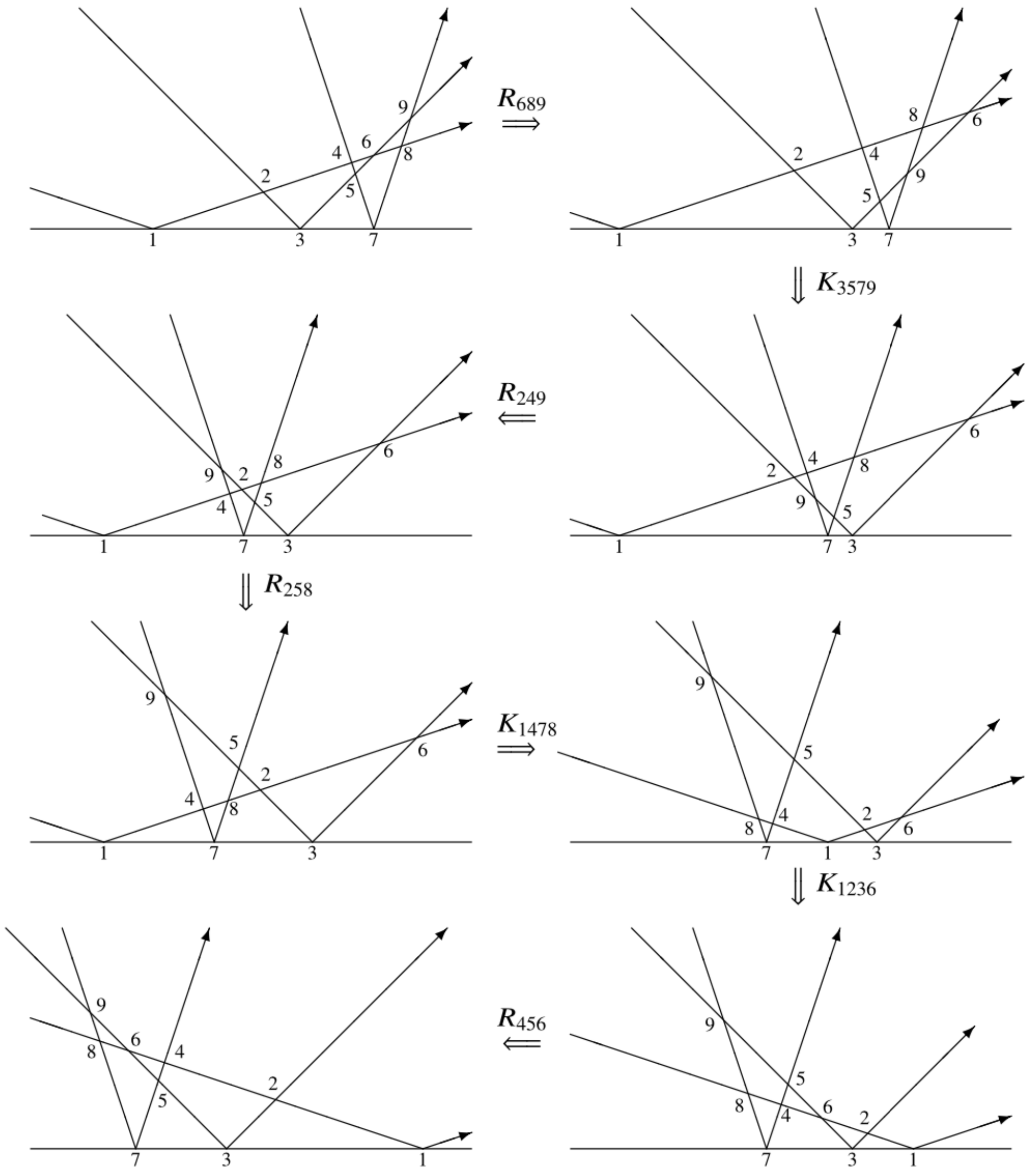


$$R_{689}K_{3579}R_{249}R_{258}K_{1478}K_{1236}R_{456} = R_{456}K_{1236}K_{1478}R_{258}R_{249}K_{3579}R_{689}$$

LHS



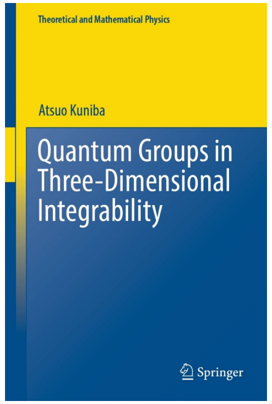
RHS



Several interesting solutions are known for the tetrahedron equation by Zamolodchikov, Baxter, Kapranov-Voevodsky, Bazhanov, Kashaev, Korepanov, Maillet, Mangazeev, Sergeev, Stroganov, Bytsko-Volkov, K-Matsuike-Yoneyama, etc.

A few solutions are known for the 3D reflection equation by K-Okado, Yoneyama.

One systematic (traditional) approach is the quantum group theoretical one using quantized coordinate rings by [Kapranov-Voevodsky 94] and PBW basis of U_q^+ by [Sergeev 08]. They are equivalent [K-Okado-Yamada 13] and have been developed extensively with many applications.



In the approach, the diagrams in the previous pages emerge as Rex (reduced expression) graphs in the Coxeter group of SL_4 and Sp_6.

The aim of this talk is to explore another approach by [Sun-Yagi 22] where these diagrams are accompanied by quivers on which the quantum cluster algebras work.

We will devise a new realization of a quantum cluster algebra by q-Weyl algebras, identify an existing solution and obtain new solutions.

2. Quantum cluster algebra [Fock-Goncharov 03,09]

Seed $= (B, \mathbf{Y})$

$B = (b_{ij})_{i,j=1}^n, \;\; b_{ij} = -b_{ji} \in \mathbb{Z}$: Exchange matrix (n fixed)

$B \leftrightarrow Q$: quiver with vertices $1, \ldots, n$

$\mathbf{Y} = (Y_1, \ldots, Y_n), \quad Y_i Y_j = q^{2b_{ij}} Y_j Y_i$: Y-variables

$i \xrightarrow{b_{ij}} j$

$\mathbb{F}(\mathbf{Y}) = \mathbb{F}(B, \mathbf{Y})$: non-commutative fraction field generated by $\mathbf{Y}$

Mutation

$\mu_k(B, \mathbf{Y}) = (B', \mathbf{Y}') \qquad k \in \{1, \ldots, n\}$

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + [b_{ki}]_+ b_{kj} + [b_{kj}]_+ b_{ik} & \text{otherwise} \end{cases} \qquad [x]_+ = \max(x, 0)$$

$$Y'_i = \begin{cases} Y_k^{-1} & i = k \\ q^{b_{ik}[b_{ki}]_+} Y_i Y_k^{[b_{ki}]_+} \prod_{m=1}^{|b_{ki}|} \left(1 + q^{-\mathrm{sgn}(b_{ki})(2m-1)} Y_k\right)^{-\mathrm{sgn}(b_{ki})} & i \neq k \end{cases}$$

μ_k on $\mathbf{Y}$ is decomposed into monomial part and dilog (automorphism) part in two $(+,-)$ ways so that the following diagram becomes commutative:

$$\begin{tikzcd} Y_i \in \mathbb{F}(\mathbf{Y}) \arrow[r, "\mu_k"] \arrow[d, mapsto] & \mathbb{F}(\mathbf{Y}) \\ Y_i' \in \mathbb{F}(\mathbf{Y}') \arrow[r, "\tau_{k,\pm}"', "\text{monomial part}" below=12pt] & \mathbb{F}(\mathbf{Y}) \arrow[u, "\mu^\sharp_{k,\pm}\ \text{dilog part}"'] \end{tikzcd}$$

$$\tau_{k,\varepsilon}(Y_i') = q^{b_{ki}[\varepsilon b_{ik}]_+} Y_i Y_k^{[\varepsilon b_{ik}]_+} \quad (\varepsilon = \pm : \textit{sign})$$

$$\mu^\sharp_{k,\varepsilon} = \mathrm{Ad}\big(\Psi_q(Y_k^\varepsilon)^\varepsilon\big), \ \text{ i.e. } \ \mu^\sharp_{k,\varepsilon}(Y_i) = \Psi_q(Y_k^\varepsilon)^\varepsilon Y_i \Psi_q(Y_k^\varepsilon)^{-\varepsilon}$$

$$\Psi_q(X) = \frac{1}{(-qX;q^2)_\infty}$$

Compositions of $\mathrm{Ad}\big(\Psi_q(Y_k^\varepsilon)^\varepsilon\big)\tau_{k,\varepsilon}$ are called cluster transformations.

Example

$$\underset{Y_1}{\overset{1}{\circ}} \longrightarrow \underset{Y_2}{\overset{2}{\circ}} \quad \xrightarrow{\mu_2} \quad \underset{Y_1(1+qY_2^{-1})^{-1}}{\overset{1}{\circ}} \longleftarrow \underset{Y_2^{-1}}{\overset{2}{\circ}} \qquad b_{12} = 1 = -b_{21},\ Y_1Y_2 = q^2Y_2Y_1$$

$$\begin{tikzcd}[column sep=small] & q^{-1}Y_1Y_2 \arrow[r, "\mu^\sharp_{2,+}"] & q^{-1}Y_1\Psi_q(q^{-2}Y_2)\Psi_q(Y_2)^{-1}Y_2 = q^{-1}Y_1(1+q^{-1}Y_2)^{-1}Y_2 \arrow[dr, equal] & \\ Y_1' \arrow[ur, "\tau_{2,+}"] \arrow[dr, "\tau_{2,-}"'] & & & Y_1(1+qY_2^{-1})^{-1} \\ & Y_1 \arrow[r, "\mu^\sharp_{2,-}"] & \Psi_q(Y_2^{-1})^{-1}Y_1\Psi_q(Y_2^{-1}) = Y_1\Psi_q(q^2Y_2^{-1})^{-1}\Psi_q(Y_2^{-1}) \arrow[ur, equal] & \end{tikzcd}$$

3. Application to the tetrahedron equation (basic idea)

Coxeter relation in the Weyl group $W(sl_4)$ in terms of indices of the simple reflections

$$\Phi: \ 121 \longleftrightarrow 212, \ 232 \longleftrightarrow 323 \quad \sigma: \ 13 \longleftrightarrow 31$$

Rex (reduced expression) graph of the longest element (indices of Φ, σ refer to the positions)

$$\begin{tikzcd}
& 123\overline{121} \arrow[r, "\Phi_{456}"] & 1\overline{232}12 \arrow[r, "\Phi_{234}"] & \overline{13}2\overline{31}2 \arrow[r, "\sigma_{12}\sigma_{45}"] & 3\overline{121}32 \arrow[r, "\Phi_{234}"] & 321\overline{232} \arrow[dr, "\Phi_{456}"] & \\
\underline{12}\overline{\underline{1}3}21 \arrow[ur, "\sigma_{34}"] \arrow[dr, "\Phi_{123}"'] & & & & & & 321323 \\
& 21\underline{232}1 \arrow[r, "\Phi_{345}"'] & 2\underline{13}2\underline{31} \arrow[r, "\sigma_{23}\sigma_{56}"'] & 23\underline{121}3 \arrow[r, "\Phi_{345}"'] & \underline{232}123 \arrow[r, "\Phi_{123}"'] & 32\underline{31}23 \arrow[ur, "\sigma_{34}"'] &
\end{tikzcd}$$

Equating the two routes

$$\sigma_{34}\Phi_{123}\Phi_{345}\sigma_{23}\sigma_{56}\Phi_{345}\Phi_{123} = \Phi_{456}\Phi_{234}\sigma_{12}\sigma_{45}\Phi_{234}\Phi_{456}\sigma_{34}$$

Formal identification $R_{ijk} = \Phi_{ijk}\sigma_{ik}$ leads to the tetrahedron equation:

$$R_{124}R_{135}R_{236}R_{456} = R_{456}R_{236}R_{135}R_{124}$$

This observation has been utilized to produce an actual solution
via quantized coordinate ring $A_q = A_q(sl_4)$ [Kapranov-Voevodsky 94]

Quantum cluster algebra				Quantized coordinate ring
Quivers attached to wiring diagram	$\leftrightarrow$	reduced expression	$\leftrightarrow$	irreducible A_q-modules
mutations cluster transformations	$\leftrightarrow$	Coxeter relation	$\leftrightarrow$	intertwiner of A_q-modules
quantum dilogarithm identity involving monomial parts	$\leftrightarrow$	loop in Rex graph	$\leftrightarrow$	tetrahedron eq.

Now this part is explained for the specific choice of Square Quiver.

In order to have a non-trivial loop, one should consider the *longest element* of the relevant Coxeter group.

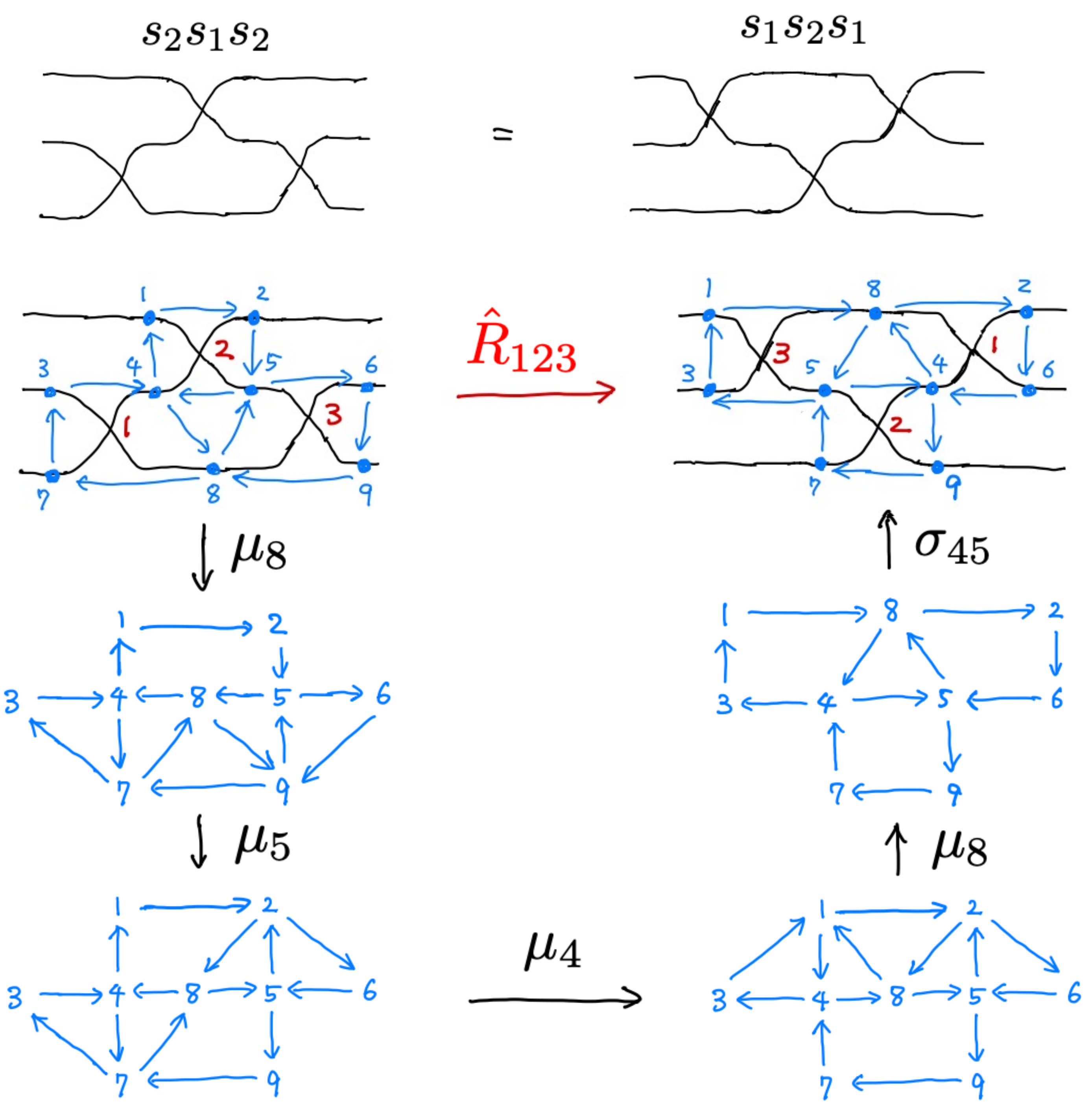
Coxeter relation
as wiring diagram

Square quiver associated
to the wiring diagram

Let $\hat{R}_{123}$ be the
cluster transformation
corresponding to

$$\sigma_{45}\mu_8\mu_4\mu_5\mu_8$$

(σ_{45} = permutation of $4, 5$)



Mutation sequence with the sign $(-,-,+,+)$ as an example:

$$(B,\mathbf{Y}) := (B^{(1)},\mathbf{Y}^{(1)}) \xrightarrow[\mu_8]{-} (B^{(2)},\mathbf{Y}^{(2)}) \xrightarrow[\mu_5]{-} (B^{(3)},\mathbf{Y}^{(3)}) \xrightarrow[\mu_4]{+} (B^{(4)},\mathbf{Y}^{(4)}) \xrightarrow[\mu_8]{+} (B^{(5)},\mathbf{Y}^{(5)}) \xrightarrow[\sigma_{45}]{} (B^{(6)},\mathbf{Y}^{(6)}) =: (B',\mathbf{Y}')$$

According to the sequence, the cluster transformation $\hat{R}_{123}: \mathbb{F}(\mathbf{Y}') \to \mathbb{F}(\mathbf{Y})$ is given by

$$\begin{aligned}\hat{R}_{123} &= \mathrm{Ad}\big(\Psi_q(Y_8^{(1)-1})^{-1}\big)\tau_{8,-}\mathrm{Ad}\big(\Psi_q(Y_5^{(2)-1})^{-1}\big)\tau_{5,-}\mathrm{Ad}\big(\Psi_q(Y_4^{(3)})\big)\tau_{4,+}\mathrm{Ad}\big(\Psi_q(Y_8^{(4)})\big)\tau_{8,+}\sigma_{45} \\ &= \mathrm{Ad}\big(\Psi_q(Y_5^{-1})^{-1}\Psi_q(Y_8^{-1})^{-1}\big) \circ \tau \circ \mathrm{Ad}\big(\Psi_q(Y_8'^{-1})\Psi_q(Y_5'^{-1})\big),\end{aligned}$$

where the monomial part $\tau = \tau_{8,-}\tau_{5,-}\tau_{4,+}\tau_{8,+}\sigma_{45}: \mathbb{F}(\mathbf{Y}') \to \mathbb{F}(\mathbf{Y})$ is given by

$$\begin{aligned}&Y_1' \mapsto Y_1, \quad Y_2' \mapsto Y_2Y_4Y_5, \quad Y_3' \mapsto q^{-1}Y_3Y_4, \quad Y_4' \mapsto qY_5^{-1}Y_8^{-1} \\ &Y_5' \mapsto Y_5, \quad Y_6' \mapsto Y_5Y_6Y_8, \quad Y_7' \mapsto qY_7Y_8, \quad Y_8' \mapsto qY_4^{-1}Y_5^{-1}, \quad Y_9' \mapsto Y_9\end{aligned}$$

Explicitly $\hat{R}_{123}$ acts as follows:

$$\begin{aligned}\alpha_4 &= 1 + q^{-1}Y_4 + Y_4Y_5, \\ \alpha_5 &= 1 + q^{-1}Y_5 + Y_5Y_8, \\ \alpha_8 &= 1 + q^{-1}Y_8 + Y_8Y_4\end{aligned}$$

$$\begin{aligned}&Y_1' \mapsto \alpha_4Y_1, \quad Y_2' \mapsto Y_2Y_4Y_5\alpha_4^{-1}, \quad Y_3' \mapsto Y_3Y_8Y_4\alpha_8^{-1}, \quad Y_4' \mapsto \alpha_4Y_5^{-1}\alpha_8^{-1}, \\ &Y_5' \mapsto \alpha_5Y_8^{-1}\alpha_4^{-1}, \quad Y_6' \mapsto \alpha_5Y_6, \quad Y_7' \mapsto \alpha_8Y_7, \quad Y_8' \mapsto \alpha_8Y_4^{-1}\alpha_5^{-1}, \quad Y_9' \mapsto Y_9Y_5Y_8\alpha_5^{-1}\end{aligned}$$

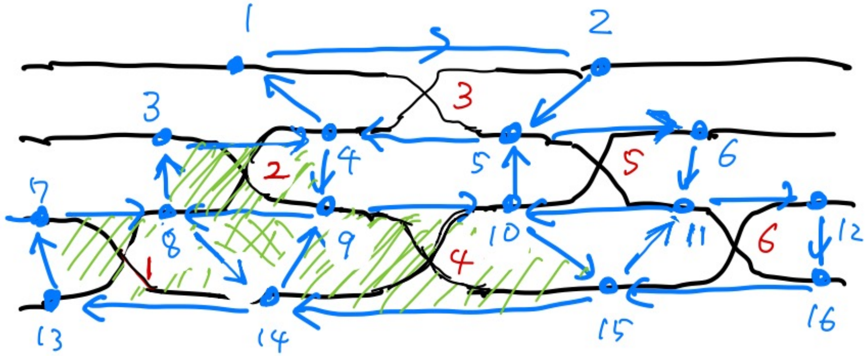
Claim: The cluster transformation $\hat{R}$ satisfies the tetrahedron equation.

Mutations/cluster transformations for RHS of the tetrahedron eq.

Black: wiring diagram, **Blue: square quiver**

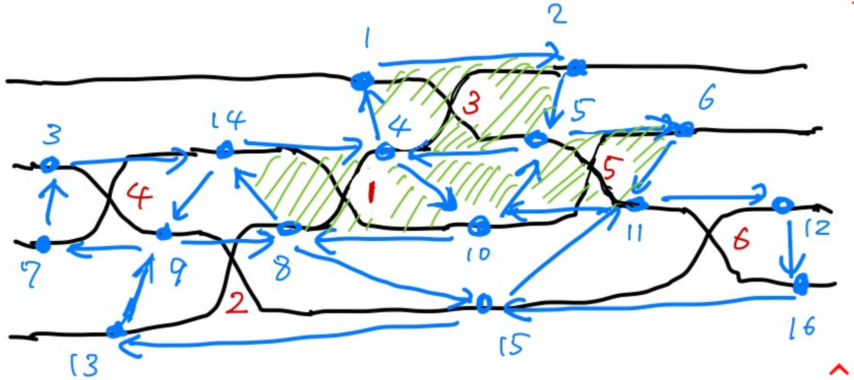
Indices of R are the vertices of the wiring diagram.

$321323 = 323123$



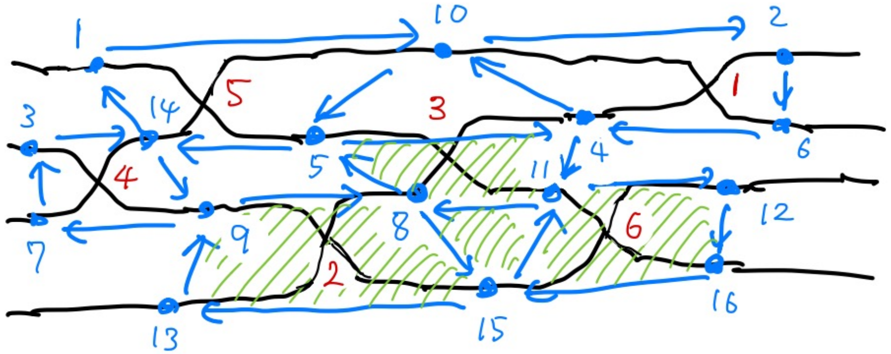
$\hat{R}_{124}$: μ_{14} μ_9 μ_8 μ_{14} $\sigma_{8,9}$

232123



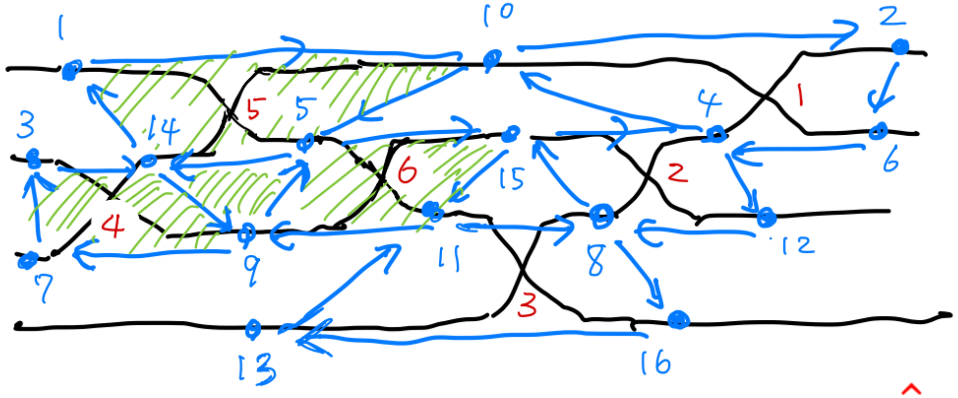
$\hat{R}_{135}$: μ_{10} μ_5 μ_4 μ_{10} $\sigma_{4,5}$

$231213 = 213231$



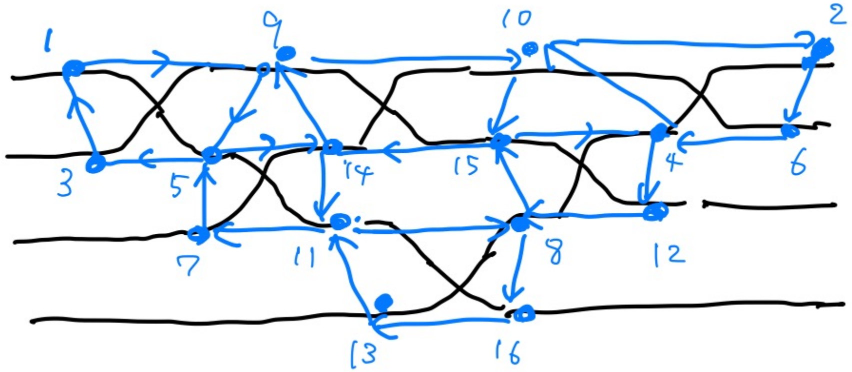
$\hat{R}_{236}$: $\sigma_{8,11}$ μ_{15} μ_8 μ_{11} μ_{15}

212321



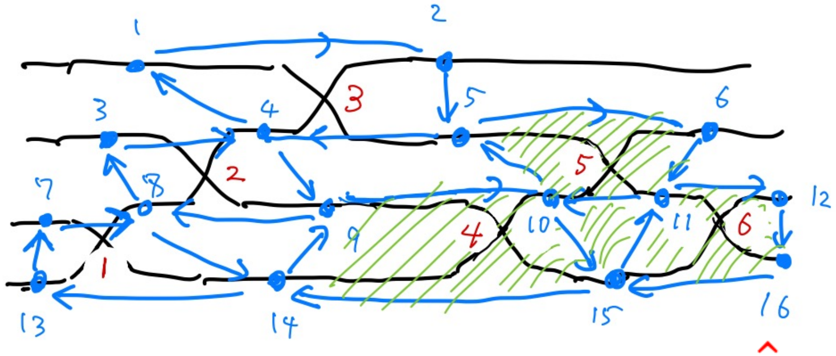
$\hat{R}_{456}$: μ_9 μ_5 μ_{14} μ_9 $\sigma_{5,14}$

121321



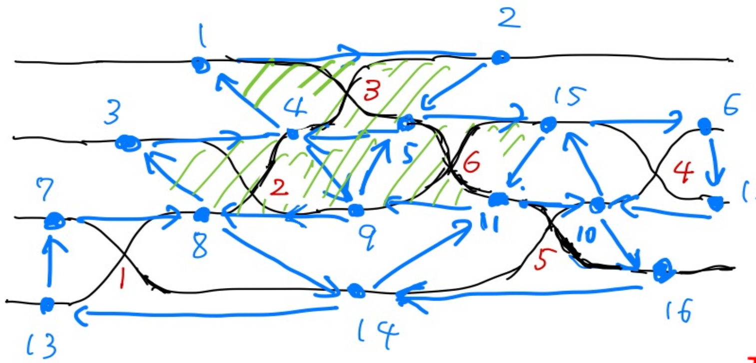
Mutations/Cluster transformations for LHS of the tetrahedron eq.

321323



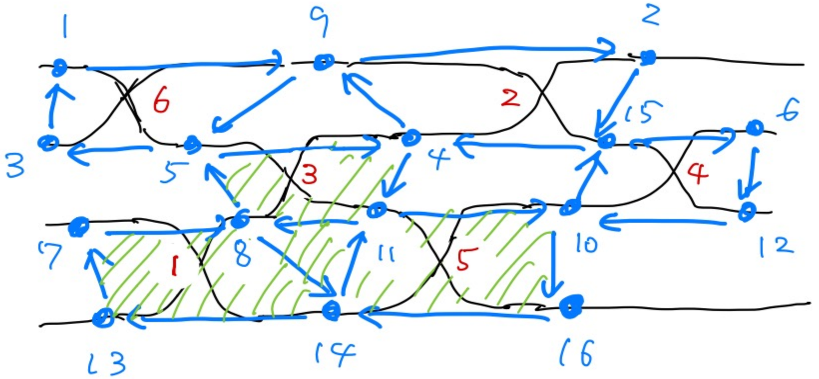
$\hat{R}_{456}$: $\mu_{15}\ \mu_{11}\ \mu_{10}\ \mu_{15}\ \sigma_{10,11}$

321232



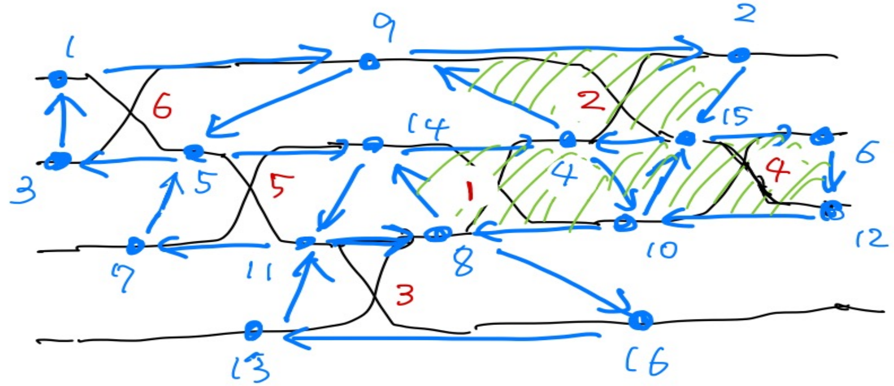
$\hat{R}_{236}$: $\mu_{9}\ \mu_{5}\ \mu_{4}\ \mu_{9}\ \sigma_{4,5}$

312132 = 132312



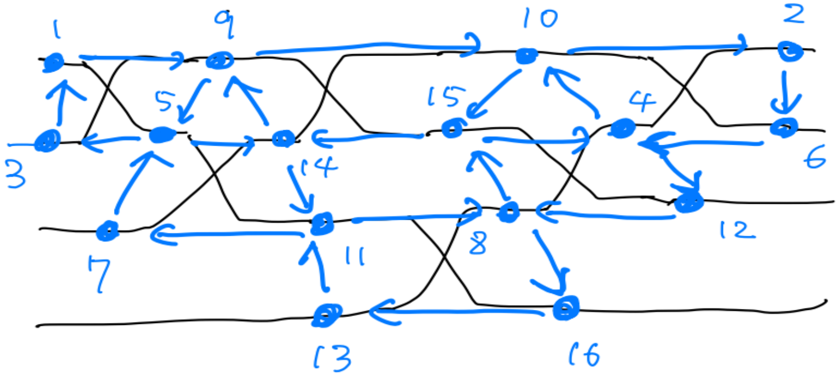
$\hat{R}_{135}$: $\sigma_{8,11}\ \mu_{14}\ \mu_{8}\ \mu_{11}\ \mu_{14}$

123212



$\hat{R}_{124}$: $\mu_{10}\ \mu_{15}\ \mu_{4}\ \mu_{10}\ \sigma_{4,15}$

123121 = 121321



= **Final quiver of RHS (previous page)**

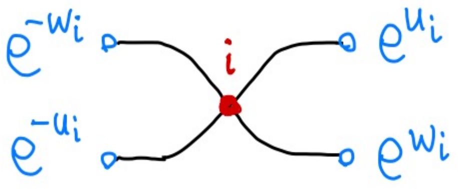
Realization in terms of q-Weyl algebra

In the initial and final wiring diagrams attach the canonical variables

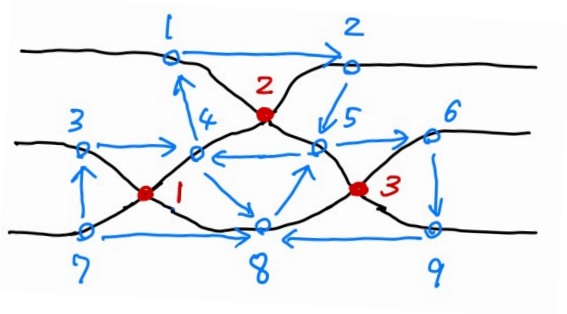
$$[u_i, w_j] = 2\hbar\delta_{ij} \quad (i, j = 1, 2, 3)$$

to the vertices 1,2,3 . $(q = e^{\hbar})$.

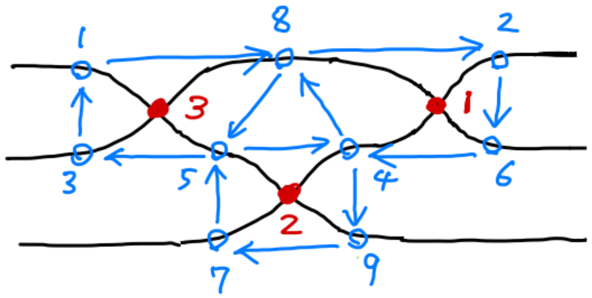
"Parameterize" the Y-variables by the rule:



Initial



Final



$$Y_1 = e^{-w_2-\lambda_2}, \quad Y_4 = e^{u_1-u_2+\lambda_1}, \quad Y_7 = e^{-u_1}$$
$$Y_2 = e^{u_2+\lambda_2}, \quad Y_5 = e^{w_2-w_3-\lambda_3}, \quad Y_8 = e^{w_1-u_3},$$
$$Y_3 = e^{-w_1-\lambda_1}, \quad Y_6 = e^{u_3+\lambda_3}, \quad Y_9 = e^{w_3}.$$

$$Y'_1 = e^{-w_3-\lambda_3}, \quad Y'_4 = e^{u_2-u_1+\lambda_2}, \quad Y'_7 = e^{-u_2},$$
$$Y'_2 = e^{u_1+\lambda_1}, \quad Y'_5 = e^{w_3-w_2-\lambda_2}, \quad Y'_8 = e^{u_3-w_1-\lambda_1+\lambda_3},$$
$$Y'_3 = e^{-u_3}, \quad Y'_6 = e^{w_1}, \quad Y'_9 = e^{w_2}.$$

$(\lambda_1, \lambda_2, \lambda_3$ are parameters for which a similar graphical rule exists.)

Proposition

(i) $Y_iY_j = q^{2b_{ij}}Y_jY_i, \quad Y'_iY'_j = q^{2b'_{ij}}Y'_jY'_i$

(ii) Monomial part τ is realized by Baker-Campbell-Hausdorff formula as

$$\tau(\cdots) = P(\cdots)P^{-1} \ \text{ with } \ P = e^{\frac{1}{2\hbar}(u_1-w_1)(w_2-w_3)}\sigma_{23}e^{\frac{\lambda_2-\lambda_3}{2\hbar}(u_3-w_1)}$$

Therefore the cluster transformation $\hat{R}_{123}$ is totally an adjoint as

$$\hat{R}_{123} = \mathrm{Ad}\Big(\Psi_q(Y_5^{-1})^{-1}\Psi_q(Y_8^{-1})^{-1}\Big)\tau\mathrm{Ad}\Big(\Psi_q(Y_8'^{-1})\Psi_q(Y_5'^{-1})\Big) = \mathrm{Ad}\big(R(\lambda_1,\lambda_2,\lambda_3)_{123}\big)$$

$$\begin{aligned} R(\lambda_1,\lambda_2,\lambda_3)_{123} &:= \Psi_q(Y_5^{-1})^{-1}\Psi_q(Y_8^{-1})^{-1}P\Psi_q(Y_8'^{-1})\Psi_q(Y_5'^{-1}) \\ &= \Psi_q(e^{w_3-w_2+\lambda_3})^{-1}\Psi_q(e^{u_3-w_1})^{-1}P\Psi_q(e^{w_1-u_3+\lambda_1-\lambda_3})\Psi_q(e^{w_2-w_3+\lambda_2}) \end{aligned}$$

(iii) Tetrahedron equation holds:

$$\begin{aligned} &R(\lambda_1,\lambda_2,\lambda_4)_{124}R(\lambda_1,\lambda_3,\lambda_5)_{135}R(\lambda_2,\lambda_3,\lambda_6)_{236}R(\lambda_4,\lambda_5,\lambda_6)_{456} \\ =\ &R(\lambda_4,\lambda_5,\lambda_6)_{456}R(\lambda_2,\lambda_3,\lambda_6)_{236}R(\lambda_1,\lambda_3,\lambda_5)_{135}R(\lambda_1,\lambda_2,\lambda_4)_{124} \end{aligned}$$

4. Relation to know solutions

R-matrix for the square quiver with sign $= (-,-,++)$

$$R(\lambda_1,\lambda_2,\lambda_3) = \Psi_q(e^{w_3-w_2+\lambda_3})^{-1}\Psi_q(e^{u_3-w_1})^{-1}P\Psi_q(e^{w_1-u_3+\lambda_1-\lambda_3})\Psi_q(e^{w_2-w_3+\lambda_2})$$

$$P = e^{\frac{1}{2\hbar}(u_1-w_1)(w_2-w_3)}\sigma_{23}e^{\frac{\lambda_2-\lambda_3}{2\hbar}(u_3-w_1)}$$

reproduces the R-matrix in “Quantum $2+1$ evolution model” [Sergeev 98], which was obtained from “face type” 3D auxiliary linear problem.

Other choices of sign provide different formulas for the same R-matrix. For example, sign $= (+,+,-,-)$ leads to

$$R(\lambda_1,\lambda_2,\lambda_3) = \Psi_q(e^{w_1-u_3})\Psi_q(e^{w_2-w_3-\lambda_3})P'\Psi_q(e^{w_3-w_2-\lambda_2})^{-1}\Psi_q(e^{u_3-w_1+\lambda_3-\lambda_1})^{-1}$$

$$P' = e^{\frac{1}{2\hbar}(w_1-u_3)(u_2+w_2)}\sigma_{13}e^{\frac{\lambda_1-\lambda_2+\lambda_3}{2\hbar}(w_1-u_3)+\frac{\lambda_1}{2\hbar}(w_2-w_3)+\frac{\lambda_3}{2\hbar}(u_1-u_2)}$$

Modular double version

Set $\hbar = i\pi b^2,\ q = e^{i\pi b^2},\ \bar{q} = e^{-i\pi b^{-2}},\ \eta = \frac{b+b^{-1}}{2}$

$$\lambda_i \to 2\pi b\lambda_i,\ u_i \to 2\pi b\hat{x}_i,\ w_i \to 2\pi b\hat{p}_i,\ [\hat{x}_j, \hat{p}_k] = \frac{i}{2\pi}\delta_{jk}$$

Non-compact quantum dilogarithm

$$\Phi_b(u) = \exp\left(\frac{1}{4}\int_{\mathbb{R}+i0} \frac{e^{-2iuw}}{\sinh(wb)\sinh(w/b)}\frac{dw}{w}\right) \qquad \frac{\Phi_b(u-ib/2)}{\Phi_b(u+ib/2)} = 1 + e^{2\pi bu} = \frac{\Psi_q(e^{2\pi b(u+ib/2)})}{\Psi_q(e^{2\pi b(u-ib/2)})} \quad \cdots (\sharp)$$

(this formula is valid when $|\mathrm{Im}\, u| < |\mathrm{Re}\, \eta|$)

From $(\sharp)$, the Modular double R acting on $L^2(\mathbb{R}^3)$ such that

(i) $\hat{x}_j$ acts as a multiplication by x_j and $\hat{p}_j$ as $-\frac{i}{2\pi}\frac{\partial}{\partial x_j}$

(ii) duality $b \leftrightarrow b^{-1}$ is implemented

is obtained by formally replacing $\Psi_q(e^{2\pi b\hat{u}})$ by $\Phi_b(\hat{u})^{-1}$:

sign $= (-,-,+,+)$:

$$\mathcal{R}(\lambda_1,\lambda_2,\lambda_3) = \Phi_b(\hat{p}_3-\hat{p}_2+\lambda_3)\Phi_b(\hat{x}_3-\hat{p}_1)\mathcal{P}\,\Phi_b(\hat{p}_1-\hat{x}_3+\lambda_1-\lambda_3)^{-1}\Phi_b(\hat{p}_2-\hat{p}_3+\lambda_2)^{-1}$$

$$\mathcal{P} = e^{2\pi i(\hat{x}_1-\hat{p}_1)(\hat{p}_3-\hat{p}_2)}\sigma_{23}e^{2\pi i(\lambda_2-\lambda_3)(\hat{p}_1-\hat{x}_3)}$$

sign $= (+,+,-,-)$:

$$\mathcal{R}(\lambda_1,\lambda_2,\lambda_3) = \Phi_b(\hat{p}_1-\hat{x}_3)^{-1}\Phi_b(\hat{p}_2-\hat{p}_3-\lambda_3)^{-1}\mathcal{P}'\,\Phi_b(\hat{p}_3-\hat{p}_2-\lambda_2)\Phi_b(\hat{x}_3-\hat{p}_1-\lambda_1+\lambda_3)$$

$$\mathcal{P}' = e^{2\pi i(\hat{x}_3-\hat{p}_1)(\hat{x}_2+\hat{p}_2)}\sigma_{13}e^{2\pi i(\lambda_1-\lambda_2+\lambda_3)(\hat{x}_3-\hat{p}_1)+\lambda_1(\hat{p}_3-\hat{p}_2)+\lambda_3(\hat{x}_2-\hat{x}_1)}$$

Integral kernel (matrix element) of the modular double R [Sergeev 10]

$$\langle x_1,x_2,x_3|x_1',x_2',x_3'\rangle = \delta(x_1-x_1')\delta(x_2-x_2')\delta(x_3-x_3')$$

$\langle x_1,x_2,x_3|\mathcal{R}(\lambda_1,\lambda_2,\lambda_3)|x_1',x_2',x_3'\rangle$ (up to normalization)

$$= \delta(x_2+x_3-x_2'-x_3')e^{2\pi i((x_3'-\lambda_1)(x_1-x_1')+(\lambda_3-i\eta)(x_2-x_1'))}\frac{\Phi_b(x_2-x_1-\lambda_1)\Phi_b(x_2'-x_1'+\lambda_2)}{\Phi_b(x_2'-x_1-i\eta)\Phi_b(x_2-x_1'-\lambda_1+\lambda_2-i\eta)}$$

“ Φ_b-analogue of the cross ratio ”

5. Outlook

$$\langle x|R|x'\rangle \sim \frac{\Phi_b(z_1)\Phi_b(z_2)\Phi_b(z_3)\Phi_b(z_4)}{\Phi_b(z_3+z_4\cdots)}$$

$(z_i = \text{linear form of } x_1,\ldots,x'_3)$

moduar double of [K-Matsuike-Yoneyama 23]

$\updownarrow$ Fourier transform

Captured by quantum cluster algebra
for the square quiver (what about the rest?)

$$\langle x|\mathcal{R}|x'\rangle \sim \delta(x_2+x_3-x'_2-x'_3) \times \frac{\Phi_b(x_2-x_1\cdots)\Phi_b(x'_2-x'_1\cdots)}{\Phi_b(x'_2-x_1\cdots)\Phi_b(x_2-x'_1\cdots)}$$

"quantum 2+1 evolution model"

[Sergeev 98, 10]

$\downarrow q^N=1$

$$R^{i_1i_2i_3}_{j_1j_2j_3} \sim \delta^{i_2+i_3}_{j_2+j_3}\frac{w_{p_1}(i_2-i_1)w_{p_2}(j_2-j_1)}{w_{p_3}(j_2-i_1)w_{p_4}(i_2-j_1)}$$

"vertex formulation of ZBB model"

[Sergeev-Mangazeev-Stroganov 95]

"vertex-IRC" duality $\longleftrightarrow$

$$\langle\sigma|R|\sigma'\rangle \sim \delta^{\sigma_1+\sigma_2}_{\sigma'_1+\sigma'_2}\delta^{\sigma_2+\sigma_3}_{\sigma'_2+\sigma'_3}\int dz\,\frac{e^{\cdots}\Phi_b(z+\frac{\sigma_1-\sigma_3\cdots}{2})\Phi_b(z+\frac{\sigma_3-\sigma_1\cdots}{2})}{\Phi_b(z+\frac{\sigma_1+\sigma_3\cdots}{2})\Phi_b(z-\frac{\sigma'_1+\sigma'_3\cdots}{2})}$$

"quantum geometry R"

[Bazhanov-Mangazeev-Sergeev 09]

$\downarrow q^N=1$

"vertex-IRC" duality $\longleftrightarrow$

$$\langle n|R|n'\rangle \sim \delta^{n_1+n_2}_{n'_1+n'_2}\delta^{n_2+n_3}_{n'_2+n'_3}\sum_{n\in\mathbb{Z}_N}\frac{q^{\cdots}w_{p_1}(n+\frac{n_1-n_3\cdots}{2})w_{p_2}(n+\frac{n_3-n_1\cdots}{2})}{w_{p_3}(n+\frac{n_1+n_3\cdots}{2})w_{p_4}(n-\frac{n'_1+n'_3\cdots}{2})}$$

"Zamolodchikov-Bazhanov-Baxter (ZBB) model"
[Bazhanov-Baxter 92]

Case of **Triangle quiver**:

A new solution to the tetrahedron and 3D reflection eqs. [Inoue-K-Terashima] in preparation

感谢您的关注

Thank you